PHYS5150 — PLASMA PHYSICS

LECTURE 7 - MAGNETIC MIRRORS

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1 MAGNETIC MIRRORS CONFIGURATION AND THE CORRESPONDING ADIABATIC INVARIANTS

We have just found that the magnetic moment μ is a constant of motion for a magnetic mirror configuration. So far we ignored possible temporal changes of the field strength. However, we have learned in the previous lecture that for each cyclic degree of freedom exists one adiabatic invariant I as long as the temporal changes are slow. So the question is how many different cyclic motion does a particle perform in a magnetic mirror configuration and what are the corresponding adiabatic invariants?

1.1 *The gyro motion around the z-axis*

First we consider the motion in the x-y-plane and assume that the B_z component is constant. This is justified as long as the changes are small compared to ω_c and $1/\rho_c$:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qv_yB_z \qquad m\frac{\mathrm{d}v_y}{\mathrm{d}t} = qv_xB_z$$

$$\ddot{x} + \omega_c^2 x = 0 \qquad \ddot{y} + \omega_c^2 y = 0.$$

This describes a harmonic oscillator and we already did this example in the previous lecture:

$$I = \pi m\omega_c\rho_c^2 \sim \frac{1}{2}|q|\omega_c\rho_c^2 \sim \mu.$$

The magnetic moment is the adiabatic invariant for the gyromotion around the z-axis.

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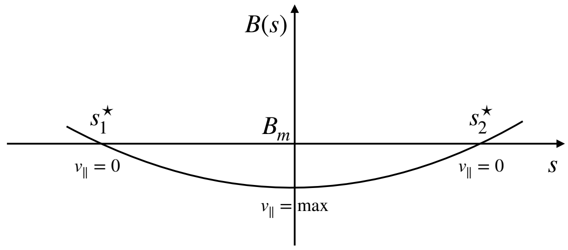
1.2 *The parallel motion between the two mirror points*



Now let us consider the particle's bouncing motion parallel to the z-axis. In the previous section we have shown that the force acting on the plasma particle in z-direction is

$$F_z = -\frac{\partial B_s}{\partial z}\mu,$$

and thus

$$-\mu\frac{\partial B}{\partial s} = m\dot{v}_\parallel = mv_\parallel\frac{\partial v_\parallel}{\partial s} = m\frac{1}{2}\frac{\mathrm{d}v_\parallel^2}{\mathrm{d}s},$$

or

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right),$$

implying that

$$W = \mu B_m = \frac{1}{2}mv_\parallel^2 + \mu B.$$

is constant and B_m is the maximum magnetic field strength. From this follows that the particle moves in an effective potential $\mu B(s)$ and

$$v_\parallel(s) = \pm\sqrt{\frac{2\mu}{m}\left(B_m - B(s)\right)}.$$

Using the definition for the adiabatic invariant we get

$$I = m\oint v_\parallel\,\mathrm{d}s = \sqrt{2\mu m}\oint\sqrt{B_m - B}\,\mathrm{d}s$$

This defines the so-called *second adiabatic invariant* J, which is associated with the periodic bouncing motion

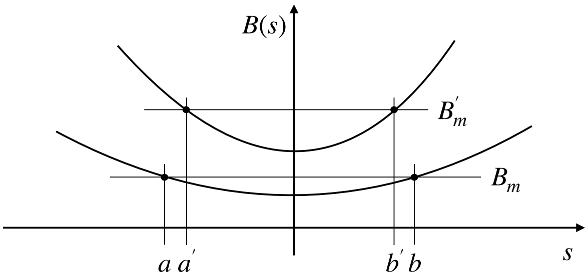
$$\boxed{J = \sqrt{2\mu m}\int_a^b\sqrt{B_m - B}\,\mathrm{d}s.} \tag{1}$$

As an application let us consider the case when the field strength of a magnetic bottle slowly changes, i.e.

$$B(s) \xrightarrow{\text{slowly}} B^{'}(s),$$

which implies that the energy is conserved. However, the second adiabatic invariant is conserved, i.e.

$$\sqrt{2\mu m}\int_a^b \sqrt{B_m - B(s)}\,\mathrm{d}s = \sqrt{2\mu m}\int_{a'}^{b'} \sqrt{B'_m - B'(s)}\,\mathrm{d}s.$$



1.3 *Fields with a pronounced axial symmetry*

Plasma particles immersed in an axially symmetric magnetic field will drift around the field axis in closed orbits. Remember that

$$\begin{aligned}
\mathbf{v}_E &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} & \mathbf{v}_E &\neq f(q,T)\\
\mathbf{v}_G &= \frac{T_\perp}{qB}\left[\frac{\hat{\mathbf{B}}\times\nabla\mathbf{B}}{B}\right] & \mathbf{v}_G &= f(q,T)\\
\mathbf{v}_c &= \frac{2T_\parallel}{qB}\left[\frac{\hat{\mathbf{B}}\times\hat{\mathbf{R}}_c}{R_c}\right] & \mathbf{v}_G &= f(q,T).
\end{aligned}$$

Consider an orbit close to the symmetry axis,

$$\int_C \mathbf{E}\cdot\mathrm{d}l = -\int_S \frac{\partial\mathbf{B}}{\partial t}\,\mathrm{d}\mathbf{A},$$

where we perform the integration along the drift contour

$$2\pi R E = -\pi R^2 \frac{\mathrm{d}B}{\mathrm{d}t},$$

or

$$E = -\frac{R}{2}\frac{\mathrm{d}B}{\mathrm{d}t}.$$

E is the azimuthal electric field, which leads to a radial $\mathbf{E} \times \mathbf{B}$ drift

$$v_E = \frac{E}{B} = -\frac{R}{2B}\frac{\mathrm{d}B}{\mathrm{d}t} \stackrel{!}{=} \frac{R}{t},$$

or

$$2\frac{\mathrm{d}R}{R} = -\frac{\mathrm{d}B}{B}.$$

From this we obtain the *third adiabatic invariant*

$$\boxed{\Phi_B = \pi R^2 B,} \tag{2}$$

which is in fact the magnetic flux.